Basic Intersection Cohomology

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- Singular Riemannian foliations
- Finiteness. Duality.
- Minimality.
- Gysin sequence.

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Introduction

• Framework : $\mathcal{F}$ singular Riemannian foliation on a manifold M.
The leaves are parallel and their dimensions may vary.

• Example : $\Phi\colon G \times M \to M$ isometric Lie group action.

$$Leaves = Orbits = G/G_x$$

• Goal : Study the cohomologies groups of M and the leaf space $M/\mathcal{F}$, and investigate their connections. Also, study the properties of this cohomology.

• Example : If $\Phi\colon \mathbb{R} \times M \to M$ is an isometric flow w/o fixed points we have

$$\cdots \longrightarrow H^k(M/\mathcal{F}) \longrightarrow H^k(M) \longrightarrow H^{k-1}(M/\mathcal{F}) \longrightarrow H^{k+1}(M/\mathcal{F}) \longrightarrow \cdots,$$

the *Gysin sequence*.

Basic Cohomology $H^*(M/\mathscr{F})$ Definition

⚠ The orbit space $M/\mathscr{F}$ can be very wild. For a flow with irrational slope on the torus T we have $T/\mathscr{F} = \mathbb{R}/\mathbb{Q}$.

- Compact case (leaves): $M/\mathscr{F}$ is a manifold, Saraké manifold, stratified pseudomanifold.

- *Basic differential forms*:

$$\Omega^*(M/\mathscr{F}) = \{\omega \in \Omega^*(M) \mid i_X\omega = i_X d\omega = 0\}$$

where X is any vector field on M tangent to $\mathscr{F}$.

- The associated cohomology is the *basic cohomology* $H^*(M/\mathscr{F})$.

- Compact case (leaves) $H^*(M/\mathscr{F}) = H^*_{sing}(M/\mathscr{F}, \mathbb{R})$.

- $\Phi\colon \mathbb{R} \times S^5 \to S^5$ $\qquad t \cdot (z_0, z_1, z_2) = (e^{ti} z_0, e^{ti} z_1, e^{ti} z_2)$

$H^*(S^5/\mathscr{F}) = \mathbb{R}[e, e^2]$ $\qquad e = d i_X \mu \in \Omega^2(S^5/\mathscr{F})$ *Euler form*

- $\Phi\colon \mathbb{R} \times S^6 \to S^6$ $\qquad x \cdot (z_0, z_1, z_2, t) = (e^{xi} z_0, e^{xi} z_1, e^{xi} z_2, t)$

$H^*(S^6/\mathscr{F}) = \mathbb{R}[e \wedge dt, e^2 \wedge dt]$

- First example: (Regular) Riemannian Foliation. Poincaré duality: $e \wedge e =$ volume form.

- Second example: Singular Riemannian Foliation. Poincaré duality: $(e \wedge dt) \wedge 0 = 0$!

Properties of the basic cohomology of a Riemannian Foliation

- $H^*(M/\mathscr{F})$ is a topological invariant. (El Kacimi-Nicolau)

- $H^*(M/\mathscr{F})$ is finite dimensional. (El Kacimi-Sergiescu-Hector)

- $H^*(M/\mathscr{F})$ verifies the Poincaré duality (El Kacimi-Hector)

- $H^n(M/\mathscr{F})$ characterizes the geometrical minimality of $\mathscr{F}$. (Masa, Álvarez)

Properties of the basic cohomology of a Singular Riemannian Foliation

- $H^*(M/\mathscr{F})$ is a topological invariant. (Wolak)

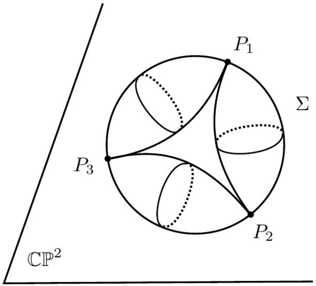
- $H^*(M/\mathscr{F})$ is finite dimensional. (Wolak)

- $H^*(M/\mathscr{F})$ does not verify the Poincaré duality.

- There do not exist minimal SRFs. (Miquel-Wolak)

Geometrical presentation of a of a singular Riemannian foliation (SRF)

- Leaves. The dimensions may vary.
- Stratification. Gluing together leaves with identical dimension. Each stratum is a (regular) Riemannian foliation.
- Example. $\Phi \colon T^2 \times \mathbb{C}P^2 \to \mathbb{C}P^2$: $(u, v) \cdot [z_1, z_2, z_3] = [uz_1, vz_2, z_3]$



Strata	
3 points	0-dimensional leaves
3 cylinders	1-dimensional leaves
$\mathbb{C}P^2 \backslash \Sigma$	2-dimensional leaves.

• Each stratum $(S, \mathscr{F})$ is a Riemannian foliation. They are arranged conically. Regular stratum (leaves of greater dimension) is dense.

• A point $x \in S \subset M$ possesses a neighborhood:

$$(\mathbb{R}^s, \mathscr{H}) \times \left(\mathrm{c}S^{m-s-1}, \mathrm{c}\mathscr{G}\right).$$

- ⋆ $(\mathbb{R}^s, \mathscr{H})$ is a foliated chart of $x \in S$ in $(S, \mathscr{F})$. Here, $s = \dim S$.
- ⋆ $\mathrm{c}S^{m-s-1}$ is the cone $S^{m-s-1} \times [0,1[/S^{m-s-1} \times \{0\}$. The apex is ϑ.
- ⋆ $\left(S^{m-s-1}, \mathscr{G}\right)$ is a SRF w/o 0-dimensional leaves. Link. Here, $m = \dim M$.

• Leaves : $H \times \{\vartheta\}$, $H \times G \times \{t\}$ $H \in \mathscr{H}, G \in \mathscr{G}, t \in]0,1[$.

• Strata : $\mathbb{R}^s \times \{\vartheta\}$, $\mathbb{R}^s \times Q \times]0,1[$ Q stratum of $\mathscr{G}$.

Basic Intersection Cohomology (BIC)

- Goal : Recover Poincaré Duality.
- Tool : Intersection Cohomology (Goresky-MacPherson) with differential forms (Brylinski).

$$\Omega^*(M/\mathscr{F}) = \{\omega \in \Omega^*(M) \mid i_X\omega = i_X d\omega = 0\}.$$

$$\Omega^*_{\overline{p}}(M/\mathscr{F}) = \{\omega \in \Omega^*((M-\Sigma)/\mathscr{F}) \mid \|\omega\| \leq \overline{p}, \|d\omega\| \leq \overline{p}\}.$$

  - ⋆ $\overline{p}$ *perversity*.
  - ⋆ $\|\omega\|$ *perverse degree*.

- Poincaré Duality

$$H^*_{\overline{p}}(M/\mathscr{F}) \cong H^{n-*}_{D\overline{p}}(M/\mathscr{F})$$

$n = \operatorname{codim}_M \mathscr{F}$, $\overline{p}$ and $D\overline{p}$ are *dual perversities*.

- A *perversity* $\overline{p}$ is a number $\overline{p}(S)$ for ech stratum S.
- *Perverse degree*. Local notion.

⋆ $(M, \mathscr{F}) = (S, \mathscr{H}) \times (\mathrm{c}S^a, \mathrm{c}\mathscr{G})$.

⋆ $\omega \in \Omega^*(S \times (S^a \backslash \Sigma) \times]0,1[)$.

⋆ $\omega(t) = \alpha_1(t) \wedge \beta_1(t) + \alpha_2(t) \wedge \beta_2(t) \wedge dt$ with $\alpha_\bullet(t) \in \Omega^*(S)$, $\beta_\bullet(t) \in \Omega^*(S^a)$.

⋆ $\|\omega\|_S = \deg \beta_1(0)$. Thom-Mather system.

Basic Intersection Cohomology (BIC) Example

$\Phi\colon \mathbb{R}\times S^6 \to S^6$ $\qquad t\cdot(z_0,z_1,z_2,x) = (e^{ti}z_0, e^{ti}z_1, e^{ti}z_2, x)$

$\Sigma = \{North, South\}$ $\qquad \mathbb{S}^6\backslash\Sigma = S^5 \times]-1,1[$

$\overline{p} \in \overline{\mathbb{Z}} = \mathbb{Z}\cup\{-\infty,\infty\}$.

$e \in \Omega^2(S^5/\mathcal{F}) \subset \Omega^2((S^6\backslash\Sigma)/\mathcal{F})$

$\|e\| = \|te\| = 2$, $\|dt\| = \|e\wedge dt\| = -\infty$, $\|1\| = \|t\| = 0$.

$H^k_{\overline{p}}(S^5/\mathcal{F})$	$\overline{-\infty}$	$\overline{p}=\overline{0}$	$\overline{p}=\overline{2}$	$\overline{p}=\overline{\infty}$
$k=0$	0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$k=1$	$\mathbb{R}[dt]$	0	0	0
$k=2$	0	0	$\mathbb{R}[e]$	$\mathbb{R}[e]$
$k=3$	$\mathbb{R}[e\wedge dt]$	$\mathbb{R}[e\wedge dt]$	0	0
$k=4$	0	0	0	$\mathbb{R}[e^2]$
$k=5$	$\mathbb{R}[e^2\wedge dt]$	$\mathbb{R}[e^2\wedge dt]$	$\mathbb{R}[e^2\wedge dt]$	0
	$H^*((S^5,\Sigma)/\mathcal{F})$	$H^*(S^5/\mathcal{F})$		$H^*((S^5\backslash\Sigma)/\mathcal{F})$

• The BIC of the foliation $\mathscr{F}$ determined by an isometric action on a compact manifold is finite dimensional.

• The BIC of the foliation $\mathscr{F}$ determined by an isometric action on a compact manifold verify the Poincaré Duality:

$$H^{*}_{\overline{p}}(M/\mathscr{F}) \cong H^{n-*}_{\overline{q}}(M/\mathscr{F})$$

with $n = \operatorname{codim} \mathscr{F}$, $\overline{p} + \overline{q} = n - 2$.

Tautness of a Riemannian foliation $(M, \mathscr{F})$ M compact

- The leaves of $\mathscr{F}$ are minimal submanifolds, relatively to a bundle-like metric μ .

- $H^n(M/\mathscr{F}) \neq 0$, under orientation hypothesis (Masa)

- $0 = [\kappa_\mu] \in H^1(M/\mathscr{F})$, Álvarez Class, (Álvarez)

(Rummler-Sullivan criterion)

Tautness of a Riemannian foliation $(S, \mathscr{X})$ S stratum of a SRF $(X, \mathscr{X})$

- The leaves of $(S, \mathscr{X})$ are minimal submanifolds, relatively to a bundle-like metric μ .

- $H_c^{n_S}(S/\mathscr{X}) \neq 0$, under orientation hypothesis, $n_S = \operatorname{codim}_S \mathscr{X}$.

- $0 = [\kappa_\mu] \in H^1(S/\mathscr{X})$, Álvarez Class

Tautness of a Singular Riemannian foliation $(X, \mathscr{X})$ X compact

- $\exists \boldsymbol{\kappa} \in H^{1}(X/\mathscr{X})$ gathering information from all strata

  - ⋆ $\boldsymbol{\kappa} = 0$ (cohomologically taut)
  - ⋆ each stratum $(S, \mathscr{X})$ is taut
  - ⋆ $(M \backslash \Sigma, \mathscr{X})$ is taut

- X simply connected $\Longrightarrow (X, \mathscr{X})$ cohomologically taut.

Gysin Sequences

- $\Phi \colon \mathbb{R} \times M \to M$ isometric action w/o fixed points

$$\cdots \longrightarrow H^k(M/\mathscr{F}) \longrightarrow H^k(M) \longrightarrow H^{k-1}(M/\mathscr{F}) \xrightarrow{\wedge[e]} H^{k+1}(M/\mathscr{F}) \longrightarrow \cdots,$$

- $\Phi \colon \mathbb{R} \times M \to M$ isometric action

$$\cdots \longrightarrow H^k(M/\mathscr{F}) \longrightarrow H^k(M) \longrightarrow H^{k-1}(M/\mathscr{F}, F) \xrightarrow{\wedge[e]} H^{k+1}(M/\mathscr{F}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H^k_{\overline{p}}(M/\mathscr{F}) \longrightarrow H^k(M) \longrightarrow H^{k-1}_{\overline{p}-\overline{2}}(M/\mathscr{F}) \xrightarrow{\wedge[e]} H^{k+1}_{\overline{p}}(M/\mathscr{F}) \longrightarrow \cdots,$$

- $\Phi\colon S^3 \times M \to M$ semi-free dim leaf $= 0, 3$

$$\cdots \longrightarrow H^k(M/S^3) \longrightarrow H^k(M) \longrightarrow H^{k-3}(M/S^3, F) \xrightarrow{\wedge[e]} H^{k+1}(M/S^3) \longrightarrow \cdots,$$

- $\Phi\colon S^3 \times M \to M$ dim leaf $= 0, 2, 3$

$$\cdots \to H^k(M/S^3) \to H^k(M) \to H^{k-3}(M/S^3, \Sigma/S^3) \oplus \left(H^{k-2}\left(M^{S^1}\right)\right)^{-\mathbb{Z}_2} \to H^{k+1}(M/S^3) \to \cdots$$

We have the Leray-Spectral sequence

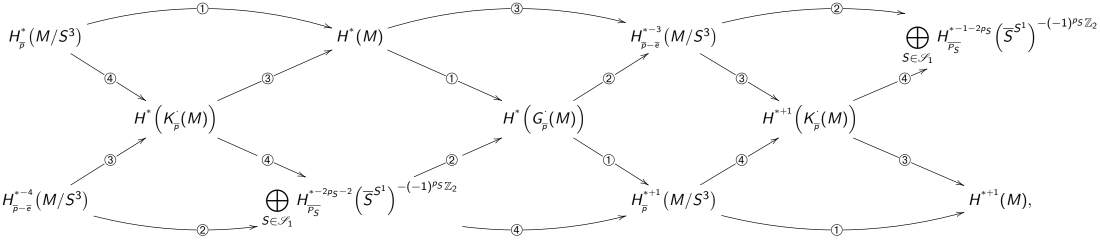
$$ {}_{\overline{p}}E_r^{i,j} \Rightarrow H^{i+j}(M). $$

The second term is given by

$$ {}_{\overline{p}}E_2^{i,j} = \begin{cases} H^i_{\overline{p}}(M/S^3) & \text{if } j = 0 \\ \bigoplus_{S \in \mathscr{S}_1} H^{i-2p_S}_{\overline{p_S}}\left(\overline{S}\right)^{-(-1)^{p_S}} & \text{if } j = 2 \\ H^i_{\overline{p}-\overline{e}}(M/S^3) & \text{if } j = 3. \end{cases} $$

It is 0 otherwise.

Gysin Braid



$H^*_{\overline{p}}(M/S^3)$
$H^*(M)$
$H^{*-3}_{\overline{p}-\overline{e}}(M/S^3)$
$\bigoplus_{S\in\mathscr{S}_1} H^{*-1-2p_S}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$
$H^*\left(K^{\cdot}_{\overline{p}}(M)\right)$
$H^*\left(G^{\cdot}_{\overline{p}}(M)\right)$
$H^{*+1}\left(K^{\cdot}_{\overline{p}}(M)\right)$
$H^{*-4}_{\overline{p}-\overline{e}}(M/S^3)$
$\bigoplus_{S\in\mathscr{S}_1} H^{*-2p_S-2}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$
$H^{*+1}_{\overline{p}}(M/S^3)$
$H^{*+1}(M),$
①
②
③
④

THANKS FOR YOUR ATTENTION !